# Edwards coordinates for elliptic curves, part 2

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# Elliptic-curve signatures

Standardize a prime $p = 2^{255} - 19$.
Not too small; want hard ECDL!
Close to $2^{\cdots}$ for fast arithmetic.

Standardize a "safe" elliptic curve $E$ over $\mathbf{F}_p$: $x^2 + y^2 = 1 + dx^2y^2$ where $d = 1 - 1/121666$.
$\#E(\mathbf{F}_p) = 8q$ where $q$ is prime.
$2(p+1) - \#E(\mathbf{F}_p) = 4 \cdot \text{prime}$.

(2005 Bernstein "Curve25519: new Diffie-Hellman speed records" as $y^2 = x^3 + 486662x^2 + x$)

Standardize $B \in E(\mathbf{F}_p)$, order $q$.
Standardize a "hash function" $H$.

Signer has 32-byte secret key $n \in \{0, 1, \ldots, 2^{256} - 1\}$.

Everyone knows signer's 32-byte public key: compressed $nB$.

To sign a message $m$:
generate a secret $s$;
compute $R = sB$;
compute $t = H(R, m)s + n \bmod q$;
transmit ($m$, compressed $R$, $t$).

To verify ($m$, compressed $R$, $t$):
verify $tB = H(R, m)R + nB$.

(first similar idea: 1985 ElGamal;
many generalizations, variations;
these choices: 2006 van Duin)

Bottleneck: Several types of elliptic-curve scalar multiplication.

Generating key:
given 256-bit integer $n$,
fixed $B \in E(\mathbf{F}_p)$, compute $nB$.

Generating signature: Same.

Verifying signature:
given 256-bit $t$, 256-bit $h$,
fixed $B$, variable $R$,
compute $tB - hR$.

Similar bottleneck for ECDH:
given 256-bit $n$, variable $R$,
compute $nR$.

# Optimizing scalar multiplication

Crypto 1985, Miller, "Use of elliptic curves in cryptography":

Using division-polynomial recursions can compute $nP$ given $P$ "in $26 \log_2 n$ multiplications"; but can do better!

"It appears to be best to represent the points on the curve in the following form: Each point is represented by the triple $(x, y, z)$ which corresponds to the point $(x/z^2, y/z^3)$."

1986 Chudnovsky/Chudnovsky, "Sequences of numbers generated by addition in formal groups and new primality and factorization tests":

"The crucial problem becomes the choice of the model of an algebraic group variety, where computations mod $p$ are the least time consuming."

For "traditional" $(X/Z^2, Y/Z^3)$:
Chudnovsky/Chudnovsky
state explicit formulas using
8**M** for DBL if $a_4 = -3$;
16**M** for ADD.

"We suggest to write
addition formulas involving
$(X, Y, Z, Z^2, Z^3)$."
9**M** DBL if $a_4 = -3$; 14**M** ADD.

Also operation counts for
projective coordinates $(X : Y : Z)$
representing $(X/Z, Y/Z)$;
Hessian curves; Jacobi quartics;
Jacobi intersections.

Asiacrypt 1998,
Cohen/Miyaji/Ono, “Efficient elliptic curve exponentiation using mixed coordinates”:

1. Faster $X, Y, Z, Z^2, Z^3$ formulas than Chudnovsky/Chudnovsky!
Compute $Z^2, Z^3$ only for points that will be added.

2. A new coordinate system; speedups in some cases.

3. A new inversion strategy.

4. The first serious analysis of parameter choices.

"Sliding windows" (1939 Brauer, improved by 1973 Thurber): popular method to compute $nP$ from $P$ using very few additions, subtractions, doublings.

Precompute $2P, 3P, 5P, 7P$.

If $n$ is even, recursively compute $(n/2)P$ and then double.

If $n$ is odd, recursively compute $(n \pm 1)P$ or $(n \pm 3)P$ or $(n \pm 5)P$ or $(n \pm 7)P$, whichever involves the largest power of 2, and then add $\mp P$ or $\mp 3P$ or $\mp 5P$ or $\mp 7P$.

Why not $2P, 3P, 5P, \ldots, 15P$?
Or $2P, 3P, 5P, \ldots, 31P$?

For $2P, 3P, 5P, \ldots, (2^w - 1)P$:
$\approx 2^{w-1}$ adds in precomputation;
on average $\approx 256/(w+2)$
adds in main computation.

Cohen/Miyaji/Ono introduce
an option to speed up the adds:
compute $2P$, convert to affine,
compute $3P, 4P$, convert,
compute $5P, 7P, 8P$, convert,
etc.

Cohen/Miyaji/Ono
analyze #adds carefully;
account for different
types of additions;
analyze several different
coordinate systems; and
identify optimal choices of $w$,
depending on **I**/**M**,
for 160 bits, 192 bits, 224 bits.

Example of results for 160 bits,
assuming $\mathbf{S}/\mathbf{M} = 0.8$:
Cohen/Miyaji/Ono recommend
one method using "1610.2**M**"
and one using "$4\mathbf{I} + 1488.4\mathbf{M}$."

Subsequent improvements:

1. Faster addition/doubling formulas for old coordinates. Many sources; for survey see Explicit-Formulas Database.

2. Fast new coordinates: e.g. Edwards curves, extended Jacobi quartics, inverted Edwards coordinates.

3. "Fractional windows" and other addition-chain tweaks: e.g. $2P, 3P, 5P, 7P, 9P, 11P, 13P$.

4. More inversion strategies.

Asiacrypt 2007, Bernstein/Lange, "Faster addition and doubling on elliptic curves": fast Edwards computations; comparison to other coordinates for scalar multiplication.

Comparison unjustifiably assumed $2P, 3P, 5P, \ldots, 15P$; ignored possibility of inversions.

New, 2007 Bernstein/Lange, "Analysis and optimization of elliptic-curve single-scalar multiplication": Much more comprehensive comparison.

Example of new results
for 160-bit scalars:
$1\mathbf{I} + 1495.8\mathbf{M}$
for Jacobian coordinates;
$1\mathbf{I} + 1434.1\mathbf{M}$
for Jacobian with $a_4 = -3$;
$1287.8\mathbf{M}$
for inverted Edwards.

Triplings? Double-base chains?
Indocrypt 2007,
Bernstein/Birkner/Lange/Peters:
triplings help Jacobian
(at least for large $\mathbf{I}/\mathbf{M}$)
but don't help Edwards.

# Many-scalar multiplication

Batch verification of many
$t_iB - h_iR_i = S_i$: check
$\sum_i v_it_iB - \sum_i v_ih_iR_i - \sum_i v_iS_i$
$= 0$ for random 128-bit $v_i$.
(Naccache et al., Eurocrypt 1994;
Bellare et al., Eurocrypt 1998)

Also encounter many scalars
in computing $nB$ as
$n_0B + n_12^{16}B + \cdots$
using precomputed $2^{16}B$ etc.

Use subtractive multi-scalar multiplication algorithm:
if $n_1 \geq n_2 \geq \cdots$ then
$n_1P_1 + n_2P_2 + n_3P_3 + \cdots = (n_1 - qn_2)P_1 + n_2(qP_1 + P_2) + n_3P_3 + \cdots$ where $q = \lfloor n_1/n_2 \rfloor$.
(credited to Bos and Coster by de Rooij, Eurocrypt 1994; see also tweaks by Wei Dai, 2007)

Addition speed is critical.
Inverted Edwards coordinates:
$9\mathbf{M} + 1\mathbf{S}$, speed record.

# Elliptic-curve factorization

Bernstein/Birkner/Lange/Peters, in progress: Edwards ECM.

First-stage ECM analysis: similar to ECC analysis. Can use larger scalars, increasing the advantage of Edwards over Montgomery.

Second stage: more complicated.

Also some improvements in curve selection.

# Elliptic-curve primality proving

Is $n$ prime? Maybe.

Want computation
of $kP$ in $E(\mathbf{Z}/n)$
to *prove* that $kP = 0$ in $E(\mathbf{Z}/p)$
for every prime divisor $p$ of $n$;
use this to prove that $n$ is prime.

Proper definition of $E(\mathbf{Z}/n)$
achieves this, but also requires
many invertibility tests,
each costing at least $1\mathbf{M}$
and extra implementation effort.

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current ECPP software
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actually prove primality?"

Morain answer to Bernstein:
"Feel free to look for a
non-prime counterexample."

Disclaimer: There is no evidence
that this conversation took place.

Often ECPP uses curves that can be transformed to Montgomery, Edwards, etc. (Chance $\to 1$ as $n \to \infty$?)

With detailed case analysis can eliminate tests for zero from a Montgomery-style ECPP. (2006 Bernstein)

Bernstein/Lange, with Jonas Lindstrøm Jensen, in progress: Aiming for simpler, faster ECPP using Edwards.